Some Characterization Of Multi-Anti Fuzzy Group

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Abstract

Theory of Multi-fuzzy set is an extension of the theory of fuzzy set, which deals with the multi-dimensional fuzziness. In this paper we extend the concept of multi-anti fuzzy subgroup and discussed some of its properties.

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1. Introduction

The theory of multi-fuzzy set was introduced by Sabu Sebastian and T.V.Ramakrishnan [23,24,25] which was in terms of multi-dimensional membership functions and they investigated some properties of multi-level fuzziness. Theory of multi-fuzzy set is an extension of theory of fuzzy sets. Complete characterization of many real life problems can be done by multi-fuzzy membership functions of the objects involved in the problem. Rosenfeld [22] started fuzzification of various algebraic concepts by his paper Fuzzy groups. N.Palaniappan and R.Muthuraj [18] introduced the inter-relationship between the anti fuzzy group and its lower level subgroups. R.Muthuraj and S.Balamurugan [15] proposed the inter-relationship between the multi-anti fuzzy group and its lower level subgroups and also they [17] proposed the inter-relationship between the multi-anti fuzzy group and its level subgroups. In this paper we extend the basic concepts of group theory into multi-fuzzy sets.

2. Preliminaries

In this section, we site the fundamental definitions that will be used in the sequel.

Definition 2.1 [28] A fuzzy subset $\mu$ of a non-empty set $X$ is a function from $X$ to $[0,1]$. That is, $\mu: X \rightarrow [0,1]$.

Definition 2.2 [23] A multi-fuzzy set $A$ of a non-empty set $X$ is

$$A = \{ (x, (\mu_i(x)))_{i \in I} : x \in X, \mu_i: X \rightarrow L_i = [0,1], i \in I \}.$$ 

The function $\mu_A = (\mu_i)_{i \in I}$ is called the multi-membership function of the multi-fuzzy set $A$. If $|I| = n$, a natural number, then $n$ is called the dimension of $A$ or $\dim(A)$.

The Complement of $A$ is $A' = \{ (x, (\mu_i'(x)))_{i \in I} : x \in X \}$, where $\mu_i'$ is the complement of $\mu_i$. That is, $\mu_i' = (1-\mu_i)$.
Definition 2.3 [25] Let $A = \{ (x, (\mu_i(x))_{i \in I}) : x \in X, i \in I, \mu_i : X \to L_i = [0,1] \}$ and $B = \{ (x, (\nu_i(x))_{i \in I}) : x \in X, i \in I, \nu_i : X \to L_i = [0,1] \}$ be multi-fuzzy sets in a non-empty set $X$. Then $A \subseteq B$ if and only if $\mu_i(x) \leq \nu_i(x)$, $\forall x \in X$ and $\forall i \in I$.

The equality, union and intersection of $A$ and $B$ are defined as:

(i) $A = B$ if and only if $\mu_i(x) = \nu_i(x)$, $\forall x \in X$ and $\forall i \in I$;

(ii) $A \cup B = \{ (x, \max \{\mu_i(x), \nu_i(x)\}_{i \in I}) : x \in X \}$;

(iii) $A \cap B = \{ (x, \min \{\mu_i(x), \nu_i(x)\}_{i \in I}) : x \in X \}$.

Proposition 2.4 [25] Let $A$ and $B$ be any two multi-fuzzy sets in $X$. Then

(i) $A \cup A = A$, $A \cap A = A$;

(ii) $A \subseteq A \cup B$, $B \subseteq A \cup B$, $A \cap B \subseteq A$ and $A \cap B \subseteq B$;

(iii) $A \subseteq B$ if and only if $A \cup B = B$;

(iv) $A \subseteq B$ if and only if $A \cap B = A$.

Proposition 2.5 If $A$ and $B$ are any two non-empty multi-fuzzy sets of a group $G$, then $(A \circ B)^{-1} = B^{-1} \circ A^{-1}$.

Proposition 2.6 [24] If $A$ and $B$ are any two multi-fuzzy sets of $X$, then

(i) $A \subseteq B$, $t = (t_1, t_2, \ldots)$ where $t_i \in [0,1] \Rightarrow A_t \subseteq B_t$, where $A_t = \{ x \in X / \mu(x) \geq t \}$

(ii) $r \leq s$, $r = (r_1, r_2, \ldots)$, $s = (s_1, s_2, \ldots)$, where $r_i, s_i \in [0,1] \Rightarrow A_r \subseteq A_s$

(iii) $A = B$ $\Leftrightarrow A_t = B_t$, $\forall t = (t_1, t_2, \ldots)$ where $t_i \in [0,1]$

Proposition 2.7 Let $\{ A_i / i \in I \}$ be the family of all multi-fuzzy sets of $G$. Then for any $t = (t_1, t_2, \ldots)$ where $t_i \in [0,1]$, $\forall i \in I$,

(i) $\cup (A_i) \subseteq (\cup A_i)_t$

(ii) $\cap (A_i) = (\cap A_i)_t$

Corollary: If $I$ is finite in the above Proposition 2.7, then equality holds in the proof (i).

3. Multi-Anti Fuzzy Group

In this section, we discuss some properties of the multi-anti fuzzy group.

Definition 3.1 [18] A fuzzy subset $\mu$ of a group $G$ is said to be anti-fuzzy subgroup of $G$ if for all $x, y \in G$,

(i) $\mu(xy) \leq \max \{\mu(x), \mu(y)\}$

(ii) $\mu(x^{-1}) = \mu(x)$
Definition 3.2 [15] A multi-fuzzy set $A$ of a group $G$ is said to be a multi-anti fuzzy subgroup of $G$ if for all $x, y \in G$,

(i) $A(xy) \leq \max\{A(x), A(y)\}$

(ii) $A(x^{-1}) = A(x)$

Definition 3.3 [15] A multi-fuzzy set $A$ of a group $G$ is called a multi-anti fuzzy subgroupoid of $G$ if:

$A(xy) \leq \max\{A(x), A(y)\}$, for all $x, y \in G$.

Theorem 3.1 The multi-fuzzy set $A = (A_1, A_2, \ldots, A_n)$ is a multi-anti fuzzy subgroup of $G$ iff each $A_i$'s are anti fuzzy subgroup of $G$, $\forall i \in I$.

Proof Let $A = (A_1, A_2, \ldots, A_n)$ be a multi-anti fuzzy subgroup of $G$.

Then $\forall x, y \in G$, $A(xy) \leq \max\{A(x), A(y)\}$

$\iff (A_i(xy))_{i \in I} \leq \max\{(A_i(x))_{i \in I}, (A_i(y))_{i \in I}\}$

$\iff (A_i(xy))_{i \in I} \leq \max\{(A_i(x)), (A_i(y))\}_{i \in I}$

$\iff A_i(xy) \leq \max\{A_i(x), A_i(y)\}$, $\forall i \in I$ and $\forall x, y \in G$

$\iff A_i$ is anti fuzzy subgroup of $G$, $\forall i \in I$

$\iff$ each $A_i$'s are anti fuzzy subgroup of $G$.

Hence the Theorem.

Theorem 3.2 The multi-fuzzy set $A = (A_1, A_2, \ldots, A_n)$ is a multi-fuzzy subgroup of a group $G$ iff its complement $A^c$ is a multi-anti fuzzy subgroup of $G$.

Proof Let $A = (A_1, A_2, \ldots, A_n)$ be a multi-fuzzy subgroup of $G$.

$\iff$ each $A_i$ is a fuzzy subgroup of $G$, by the Theorem 3.1.

$\iff$ each $A_i^c$ is anti fuzzy subgroup of $G$

$\iff (A_1^c, A_2^c, \ldots, A_n^c)$ is a multi-anti fuzzy subgroup of $G$, by the Theorem 3.1.

$\iff A^c$ is a multi-anti fuzzy subgroup of $G$.

Definition 3.4 Let $A$ be a multi-fuzzy subset of a group $G$ and let $\langle A \rangle = \bigcup\{B \mid A \subseteq B, B$ is a multi-anti fuzzy subgroup of $G\}$. Then $\langle A \rangle$ is called the multi-anti fuzzy subgroup of $G$ generated by $A$.

Theorem 3.3 Let $A$ be a multi-fuzzy set of $G$ and let $\langle A \rangle = \bigcup\{B \mid A \subseteq B, B$ is a multi-anti fuzzy subgroup of $G\}$. Prove that $\langle A \rangle$ is the multi-anti fuzzy subgroup of $G$.

Proof Let $x, y \in G$.

Then $\langle A \rangle(xy^{-1}) = (\bigcup B)(xy^{-1})$
\[
\text{Sup } B(xy^{-1}) \\
\leq \text{Sup } \{ \text{max} \{B(x), B(y)\} \} \text{, since } B \text{ is a multi-anti fuzzy subgroup of } G.
\]
\[
= \max \{ \text{Sup } B(x), \text{Sup } B(y)\}
\]
\[
= \max \{ (\cup B)(x), (\cup B)(y)\}
\]
\[
= \max \{ \langle A \rangle(x), \langle A \rangle(y) \}
\]
That is, \(\langle A \rangle(xy^{-1}) \leq \max \{ \langle A \rangle(x), \langle A \rangle(y) \}\)

Therefore, \(\langle A \rangle\) is a multi-anti fuzzy subgroup of \(G\).

Remarks
1. \(\langle A \rangle\) is called the multi-anti fuzzy subgroup of \(G\) generated by \(A\).
2. \(\langle A \rangle\) is the largest multi-anti fuzzy subgroup of \(G\) which contains \(A\).

Theorem 3.4 Let \(H\) be a group and \(B\) be a multi-anti fuzzy subgroup of \(H\). Let \(f\) be a homomorphism of \(G\) into \(H\). Then \(f^{-1}(B)\) is a multi-anti fuzzy subgroup of \(G\).

Proof Let \(x, y \in G\) and let \(B = (B_1, B_2, ..., B_i, ..., B_n)\) be a multi-anti fuzzy subgroup of \(H\).

\[\implies\] each \(B_i\) ‘s are anti fuzzy subgroup of \(H\), by the Theorem 3.1.

Given \(f : G \rightarrow H\) is a homomorphism.

Claim: \(f^{-1}(B)\) is a multi-anti fuzzy subgroup of \(G\). i.e., \(f^{-1}(B_i)\) is anti fuzzy subgroup of \(G\).

Now, \(f^{-1}(B_i)(xy^{-1}) = B_i(f(xy^{-1}))\), by the extension principle of fuzzy set.

\[
= B_i(f(x)f(y^{-1})) \text{, since } f \text{ is homomorphism.}
\]
\[
= B_i(f(x)f(y))
\]
\[
\leq \max \{ B_i(f(x)), B_i(f(y)) \} \text{, since each } B_i \text{ is anti fuzzy subgroup of } H.
\]
\[
= \max \{ B_i(f(x)), B_i(f(y)) \}
\]
\[
= \max \{ f^{-1}(B_i)(x), f^{-1}(B_i)(y) \}
\]
Therefore, \(f^{-1}(B_i)\) is anti fuzzy subgroup of \(G\), \(\forall i\).

Which implies that \(f^{-1}(B)\) is a multi-anti fuzzy subgroup of \(G\), by the Theorem 3.1.

Hence the Theorem.

Theorem 3.5 If both \(A\) and \(B\) are multi-anti fuzzy subgroups of \(G\), then \((A+B)\) is a multi-anti fuzzy subgroup of \(G\).

Proof Given that both \(A\) and \(B\) are multi-anti fuzzy subgroups of \(G\).

That is, \(\forall x, y \in G\), \(A(xy^{-1}) \leq \max \{ A(x), A(y) \}\) and \(B(xy^{-1}) \leq \max \{ B(x), B(y) \}\)

Now, \(\forall x, y \in G\),

\[
(A+B)(xy^{-1}) = A(xy^{-1}) + B(xy^{-1}) - A(xy^{-1})B(xy^{-1})
\]
\[ \leq \max\{ A(x), A(y) \} + \max\{ B(x), B(y) \} - \max\{ A(x), A(y) \} \max\{ B(x), B(y) \} \]
\[ = \max\{ A(x) + B(x) - A(x)B(x), A(y) + B(y) - A(y)B(y) \} \]
\[ = \max\{ (A+B)(x), (A+B)(y) \} \]

That is, \((A+B)(xy^{-1}) \leq \max\{ (A+B)(x), (A+B)(y) \}, \forall x,y \in G\)

Therefore, \((A+B)\) is a multi-anti fuzzy subgroup of \(G\).

Hence the Theorem.

**Theorem 3.6** If \(A\) and \(B\) are multi-anti fuzzy subgroups of \(G\), then \((A+B)\) is a multi-anti fuzzy subgroup of \(G\).

**Proof** It is clear.

**Theorem 3.7** Let \(G\) be a group and \(A\) be a multi-anti fuzzy subgroup of \(G\). If \(A(x) < A(y)\) for some \(x,y \in G\), then \(A(xy) = A(y) = A(yx)\).

**Proof** Given that \(A(x) < A(y)\) for some \(x,y \in G\).

Since \(A\) is a multi-anti fuzzy subgroup of \(G\),

\[ \forall x,y \in G, \ A(xy) \leq \max\{A(x), A(y)\} = A(y) \ldots \ldots \ldots (1) \]

Now, \(A(y) = A(x^{-1}(xy)) \)
\[ \leq \max\{A(x^{-1}), A(xy)\} \]
\[ = \max\{A(x), A(xy)\} \]
\[ = A(xy), \text{ since } A(x) < A(y), A(x) < A(xy) \]

That is, \(A(y) \leq A(xy) \ldots \ldots \ldots \ldots \ldots (2) \)

Because, \(A(x) < A(y) \Rightarrow A(xx) < A(xy)\)

but \(A(xx) \leq \max\{A(x), A(x)\} = A(x)\)

That is, this \(\Rightarrow A(xx) \leq A(x) \text{ and } A(x) < A(y)\)

That is, \(A(xx) \leq A(x) < A(y)\)

That is, \(A(xx) \leq A(x) < A(xy) \leq A(y), \text{ since by (1)}\)

Therefore, \(A(x) < A(xy)\).

From (1) and (2), we get \(A(xy) = A(y)\).

Similarly, we can prove \(A(yx) = A(y)\).

Hence the Theorem.
Theorem 3.8 Let G be any group of prime order with identity ‘e’ and let A be any multi-anti fuzzy subgroup of G. Then prove A(x)=A(y)≥A(e), ∀x,y∈G-{e}. Conversely, any such A will be a multi-anti fuzzy subgroup of G.

Proof Given that o(G) = p , prime number.
That is, ∀x∈G, x^p=e.
Let x,y∈ G-{e} ⇒ x(≠e), y(≠e)∈G
⇒x^p=e ; y^p=e
⇒x^p = y^p
⇒ x = y
⇒ A(x) = A(y) ......................(1)
Since A is a multi-anti fuzzy subgroup of G, ∀x,y∈G,
A(e) ≤ A(x) and A(e) ≤ A(y)
From (1), we get A(e) ≤ A(x) = A(y)
Hence, A(x) = A(y)≥A(e), ∀x,y∈G-{e} and hence the proof.
Conversely, Suppose A(x) = A(y) ≥A(e), ∀x,y∈G-{e}.
Claim: A is a multi-anti fuzzy subgroup of G .
That is, A(xy^{-1}) ≤ max{A(x), A(y)} , ∀x,y∈G.
Now, ∀x,y∈G-{e} ⇒ x,y^{-1}∈G-{e}, since G is a group
⇒ xy^{-1}∈G, since G is a group
⇒ (xy^{-1})^p = e , since o(G)=p
⇒ (xy^{-1})^p = e^p , since e^p = e
⇒ xy^{-1} = e  .........................(1)
⇒ (xy^{-1})y= (e)y
⇒ x = y
⇒ A(x) = A(y)
⇒ A(x) = A(y)≥A(e) , by the hypothesis
Now, , ∀x,y∈G-{e}, A(xy^{-1}) = A(e) , since by (1)
≤ A(x) = A(y) , by the hypothesis
= max{A(x), A(y)}
That is, A(xy^{-1}) ≤ max{A(x), A(y)}, ∀x,y∈G-{e}
Therefore, A(xy^{-1}) ≤ max{A(x), A(y)}, ∀x,y∈G
That is, A is a multi-anti fuzzy subgroup of G.
**Definition 3.5** A multi-fuzzy set A of a set G is said to have “inf property” if, for any subset $G_1$ of G, there exists $a_0 \in G_1$ such that $A(a_0) = \inf \{A(a) / a \in G_1\}$.

**Theorem 3.9** Let A be a multi-anti fuzzy subgroup of a group G with identity ‘e’. If $A_t$, where $t = A(e)$, has finite index, then A has inf property.

**Proof** Given $A_t$ has finite index and $t = A(e)$.

By the definition, $A_t = \{ x \in G / A(x) \leq t = A(e) \}$.

Since A is a multi-anti fuzzy subgroup of G, $A(e) \leq A(x), \forall x \in G$.

Therefore (1) becomes as, $A_t = \{ x \in G / A(x) = A(e) \}$

Claim: A has inf property

Consider $\inf A(x)$, since $A_t$ has finite index

$x \in A_t$

$= \inf A(e)$, since $x \in A_t$, $A(x) = A(e)$

$x \in A_t$

$= A(e)$, since $e \in A_t$, as $e \in G$

That is, there exists $e \in A_t$ such that $A(e) = \inf A(x)$

$x \in A_t$

Therefore, by the definition 3.5, A has inf property and hence the Theorem.

**Definition 3.6** The family of lower level subgroups of A is $\{A_t / t \in \text{Im } A\}$ where A is any multi-anti fuzzy subgroup of a group G and it is denoted by $F_A$. That is, $F_A = \{A_t / t \in \text{Im } A\}$.

Moreover, if $\text{Im } A = \{t_0, t_1, t_2, ..., t_n\}$ with $t_0 < t_1 < t_2 < ...... < t_n$, then the lower level subgroups of A form the chain: $A_{t_0} \subset A_{t_1} \subset A_{t_2} \subset ...... \subset A_{t_n} = G$.

**Theorem 3.10** Two multi-anti fuzzy subgroups A and B such that $\text{Card } \text{Im } A < \infty$ and $\text{Card } \text{Im } B < \infty$, of a group G are equal $\iff \text{Im } A = \text{Im } B$ and $F_A = F_B$.

**Proof** Given A and B are two multi-anti fuzzy subgroups of a group G and $A = B$.

$A = B \iff A(x) = B(x), \forall x \in G$

$\iff \text{Im } A = \text{Im } B$

$\iff \{t_0, t_1, ..., t_n\} = \{s_0, s_1, ..., s_n\}$ with $t_0 < t_1 < ...... < t_n$; $s_0 < s_1 < ...... < s_n$ and $t_i = s_i, i=0,1,..,n$, since $\text{Card } \text{Im } A < \infty$ and $\text{Card } \text{Im } B < \infty$. 
∀i=0,1,…,n, clearly Aₙ and Bₙ are level subsets of A and B respectively ↔ Aₙ and Bₙ are lower level subgroups of A and B respectively, since A and B are multi-anti fuzzy subgroups of G.

Consider for i=0,1,2,…,n,

\[ t_i = s_i \Leftrightarrow A_{t_i} = B_{s_i}, \forall i=0,1,\ldots,n \], since A=B.

\[ \Leftrightarrow \{ A_{t_i} / t_i \in \text{Im } A \} = \{ B_{s_i} / s_i \in \text{Im } B \}, \forall i=0,1,\ldots,n. \]

\[ \Leftrightarrow F_A = F_B, \text{ since by the definition 3.6.} \]

Hence the proof of the Theorem.

**Theorem 3.11** Let A be any multi-anti fuzzy subgroup of a group G with identity ‘e’. Then Prove that A(xy)=A(y), ∀y∈G ↔ A(x)=A(e) where x∈G.

**Proof** Suppose A(xy)=A(y), ∀ y∈G

\[ \Rightarrow A(xe) = A(e), \text{ since } e \in G \]

\[ \Rightarrow A(x) = A(e), \text{ since } xe = x \in G \text{ as } G \text{ is a group} \]

Conversely, Suppose A(x)=A(e) where x∈G and let y∈G.

Since A is a multi-anti fuzzy subgroup of G, A(xy)≤max{A(x),A(y)}

That is, A(xy)≤max{A(e),A(y)}, since by the hypothesis.

\[ = A(y), \text{ since A is a multi-anti fuzzy subgroup of G, A(e)≤A(x), } \forall x \in G \]

That is, A(xy)≤A(y), ∀y∈G ……………………………(1)

For all y∈G,

\[ A(y) = A((x^{-1}x)y) \text{ where } x \in G \]

\[ \leq \max \{ A(x^{-1}), A(xy) \} \]

\[ = \max \{ A(x), A(xy) \} \]

\[ = \max \{ A(e), A(xy) \}, \text{ since by the hypothesis} \]

\[ = A(xy), \text{ since } A(e)\leq A(xy), \text{ as } xy \in G \text{ and } A \text{ is a multi-anti fuzzy subgroup} \]

That is, A(y)≤A(xy), ∀y∈G ……………………………(2)

Therefore, from (1) and (2) we get A(xy)=A(y), ∀ y∈G

Hence the proof of converse part and hence the Theorem.

**Theorem 3.12** If A is a multi-anti fuzzy subgroupoid of a finite group G, then A is a multi-anti fuzzy subgroup of G.

**Proof** Since A is a multi-anti fuzzy subgroupoid of G,

\[ A(xy) \leq \max \{ A(x), A(y) \}, \forall x,y \in G. \]

Claim: A is a multi-anti fuzzy subgroup of G
That is, to prove $A(x^{-1})=A(x), \forall x \in G$

Since $A$ is a multi-fuzzy set of $G$, by Theorem,

$$A = A^{-1}$$

$$\Rightarrow A(x) = A^{-1}(x), \forall x \in G$$

$$\Rightarrow A(x) = A(x^{-1}), \forall x \in G$$, since $A$ is a multi-anti fuzzy subgroup of $G$.

Hence the Theorem.

**Theorem 3.13** Let $C$ be any multi-anti fuzzy subgroup of $G$ such that $\text{Im}(C) = \{ t=(t_1, t_2, \ldots, t_n) / t_i \in [0,1], i=1,2,\ldots,n \}$. If $C = A \cap B$, where $A$ and $B$ are multi-anti fuzzy subgroups of $G$, then either $A \subseteq B$ or $B \subseteq A$.

**Proof** If possible, let $A(x) > B(x)$ and $B(y) > A(y)$, for some $x, y \in G$.

Since $\tau \in \text{Im}(C)$, then $\tau = C(x) = (A \cap B)(x) = \min \{A(x), B(x)\} = B(x) < A(x)$ \ldots (1)

and $\tau = C(y) = (A \cap B)(y) = \min \{A(y), B(y)\} = A(y) < B(y)$ \ldots (2)

Therefore, from combining (1) and (2) we get,

$$B(y) > \tau = B(x) \text{ and } A(x) > \tau = A(y) \ldots \ldots \ldots \ldots (3)$$

That is, $B(x) < B(y)$ and $A(y) < A(x)$

This $\Rightarrow B(xy) = B(y)$ and $A(xy) = A(x)$, by the Theorem 3.7 \ldots (4)

But, then $\tau = C(xy) = (A \cap B)(xy) = \min \{A(xy), B(xy)\}$

$$= \min \{A(x), B(y)\}$$, by using (4)

$$> \tau$$, since by (3)

That is, $\tau > \tau$, which is untenable.

Hence the Theorem.

**Theorem 3.14** If $A$ is a multi-anti fuzzy subgroup of $G$ and let $A_\ast = \{ x \in G / A(x) = A(e) \}$, prove that $A_\ast$ is a subgroup of $G$.

**Proof** It is clear.

**Theorem 3.15** If $A$ is a multi-anti fuzzy subgroup of $G$ and let $A^* = \{ x \in G / A(x) > 0 \}$, then prove the Support of $A$, $A^*$ is a subgroup of $G$.

**Proof** It is clear.

**Theorem 3.16** If $A$ is any multi-anti fuzzy subgroup of a group $G$, then $A^{-1} = A$.

**Proof** It is clear.
Theorem 3.17 Let $C$ be any multi-anti fuzzy subgroup of $G$ such that $\text{Im}(C) = \{0,t\}$, where $t=(t_1,t_2,\ldots,t_n)$, $t_i \in [0,1]$, $\forall i=1,2,\ldots,n$. If $C=A\cap B$, for some multi-anti fuzzy subgroups $A$ and $B$ of $G$, then either $A \subseteq B$ or $B \subseteq A$.

Proof To obtain a proof by contradiction, Assume that $A(x) > B(x)$ and $B(y) > A(y)$ for some $x,y \in G$.

Then, since $\text{Im}(C)=\{0,t\}$, $t = C(x)$
\[ = (A \cap B)(x) \]
\[ = \min\{A(x),B(x)\} \]
\[ = B(x) \]
\[ < A(x) \] (1) and
\[ t = C(y) \]
\[ = (A \cap B)(y) \]
\[ = \min\{A(y),B(y)\} \]
\[ = A(y) \]
\[ < B(y) \] (2)

Therefore, from combining (1) and (2), we get
\[ B(x) = t < B(y) \text{ and } \]
\[ A(y) = t < A(x) \] (3)

That is, $B(x) < B(y)$ and $A(y) < A(x)$.

Hence, by the Theorem 3.7, $A(xy) = A(x)$ and $B(xy) = B(y)$ (4)

Hence, $A(xy) = A(x) > t$ and $B(xy) = B(y) > t$, from (3) and (4) (5)

Now, $C(y) = (A \cap B)(y)$
\[ = \min\{A(y),B(y)\} \]
\[ = A(y) \]
\[ = t \]
\[ = B(x) \]

Therefore, this $\Rightarrow B(x) = C(y)$ (6)

Therefore, $t = C(x) = B(x)$, since by (1)
\[ = C(y) \text{, since by (6)} \]
\[ = A(y) \text{, since by (2) ( since Im(C) = \{0,t\} )} \]

Hence, $C(xy) = (A \cap B)(xy)$
\[ = \min\{A(xy),B(xy)\} \]
\[ > t \text{, since from (5)} \]

That is, $C(xy) > t = \max\{C(x),C(y)\}$, since $C(x) = C(y) = t$

\[ \geq C(xy) \text{, since C is a multi-anti fuzzy subgroup of G} \]
Therefore, this \( \Rightarrow C(xy) > C(xy) \), which is the desired contradiction.
Hence the Theorem also.

**Theorem 3.18** A multi-fuzzy subset \( A \) of a group \( G \) is a multi-anti fuzzy subgroup of \( G \) if and only if each multi-lower level subset \( A_t \) where \( t \in \text{Im}(A) \), are subgroups of \( G \).

**Proof** It is clear.

**Theorem 3.19** If \( C \) is a multi-anti fuzzy subgroup of \( G \) with \( 3 \leq \text{card} \text{Im}(C) < \infty \), then there exist multi-anti fuzzy subgroups \( A \) and \( B \) of \( G \) satisfying \( C = A \cap B, A \not\subset B \) and \( B \not\subset A \).

**Proof** Let \( \text{Im}(C) = \{ t_i = (t_{i1}, t_{i2}, ..., t_{im}) \mid 2 \leq n \leq \infty, m < \infty \} \) and \( t_0 < t_1 < ... < t_n \).
Choose \( s_i = (s_{i1}, s_{i2}, ..., s_{im}) \), \( i = 1, 2 \) where \( s_i \in [0,1] \), \( j = 1, 2, ..., m \) to be such that \( t_0 < s_1 < t_1 < s_2 < t_2 < ... < t_n \).
Then \( \{ e \} \subset C_{t_0} \subset C_{t_1} \subset C_{t_2} \subset C_{t_3} \subset ... \subset C_{t_n} = G \).
That is, multi-lower level subgroups of a multi-anti fuzzy subgroup \( C \) of a group \( G \) form the chain.
Define multi-fuzzy subsets \( A \) and \( B \) of \( G \) by

\[
A(x) = \begin{cases} 
  s_2 & \text{if } x \in C_{t_0} \\
  t_0 & \text{if } x \in C_{t_1} \\
  C(x) & \text{otherwise}
\end{cases} \quad \text{and} \quad B(x) = \begin{cases} 
  t_1 & \text{if } x \in C_{t_0} \\
  s_1 & \text{if } x \in C_{t_1} \\
  C(x) & \text{otherwise}
\end{cases}
\]

It is clear from Theorem 3.18 that, \( A \) and \( B \) are multi-anti fuzzy subgroups of \( G \).
Finally, a routine computation confirms that \( C = A \cap B, A \not\subset B \) and \( B \not\subset A \).
Hence the Theorem.

**Theorem 3.20** If \( C \) is a multi-anti fuzzy subgroup of \( G \) such that \( \text{Im}(C) = \{ t_0, t_1 \} \) where \( t_i = (t_{i1}, t_{i2}, ..., t_{im}) \), \( i = 0, 1 \);
\( t_0 \in [0,1], t_1 \in [0,1], j = 1, 2, ..., m \) and \( t_0 < t_1 \), then there exist multi-anti fuzzy subgroups \( A \) and \( B \) of \( G \) such that
\( C = A \cap B, A \not\subset B \) and \( B \not\subset A \).

**Proof** It is clear.

**Theorem 3.21** Let \( A \) be a multi-anti fuzzy subgroup of a group \( G \). Then prove

i. the identity element of \( A \) is unique and
ii. the inverse of any element of \( A \) is unique.

**Proof** Let \( A(e_1) \) and \( A(e_2) \) be two identity elements of multi-anti fuzzy subgroup \( A \).
Since \( A(e_1) \) is an identity element of \( A \),
\( A(e_1) \leq A(e_2) \) \((1)\)
Since $A(e_2)$ is another identity element of $A$,

$$A(e_2) \leq A(e_1) \quad \text{(2)}$$

Therefore, from (1) and (2), we get

$$A(e_1) \leq A(e_2) \leq A(e_1)$$

This implies that $A(e_1) = A(e_2)$

Hence the identity element of $A$ is unique.

Let $x'$ and $x^\ast$ be two inverses of $x \in G$.

Then $xx' = e$ and $xx^\ast = e$, since $G$ is a group.

This implies that $A(xx') = A(e)$ and $A(xx^\ast) = A(e)$

$$\implies A(x^{-1}x') = A(x^{-1}e) \quad \text{and} \quad A(x^{-1}x^\ast) = A(x^{-1}e)$$

$$\implies A(ex') = A(x^{-1}) \quad \text{and} \quad A(ex^\ast) = A(x^{-1})$$

$$\implies A(x') = A(x^{-1}) \quad \text{and} \quad A(x^\ast) = A(x^{-1})$$

$$\implies A(x') = A(x^\ast)$$

$$\implies$$ inverse is unique in $A$.

Hence the Theorem.

**Theorem 3.22** If $A$ is a multi-anti fuzzy subgroup of a group $G$ and $A(xy^{-1}) = A(e)$, then for all $x, y \in G$,

1. $A(x) = A(y^{-1})$,
2. $A(x^{-1}) = A(y^{-1})$
3. $A(x^{-1}) = A(y)$

**Proof** It is clear.

**Theorem 3.23** Let $A$ be a multi-anti fuzzy subgroup of a group $G$. Then $A(x^{-1}) = A(x)$ and $A(e) \leq A(x)$, $\forall x \in G$, where ‘$e$’ is the identity element of $G$.

**Proof** $\forall x \in G, A(x) = A((x^{-1})^{-1})$

$$\geq A(x^{-1})$$

$$\geq A(x)$$

Hence $A(x) = A(x^{-1})$, $\forall x \in G$………(1)

Now, $A(e) = A(xx^{-1})$

$$\leq \max\{A(x), A(x^{-1})\}$, since $A$ is a multi-anti fuzzy subgroup of $G$.

$$= \max\{A(x), A(x)\}$, since by (1)

$$= A(x)$$

That is, $A(e) \leq A(x), \forall x \in G$. 
**Theorem 3.24**[15] If $A$ is a multi-anti fuzzy subgroup of a group $G$ with identity ‘e’, then prove that $A(xy^{-1})=A(e) \Rightarrow A(x)=A(y), \forall x, y \in G$.

**Proof** It is clear.

**Theorem 3.25**[15] $A$ is a multi-anti fuzzy subgroup of a group $G \iff A(xy^{-1}) \leq \max\{A(x), A(y)\}, \forall x, y \in G$.

**Proof** It is clear.

**Theorem 3.26** Let $\overline{A}$ be the collection of all multi-anti fuzzy subgroups of a group $G$ and $\overline{B}$ be the collection of all multi-lower level subgroups of members of $\overline{A}$. Then there is a one-one correspondence between the subgroups of $G$ and the equivalence classes of multi-lower level subgroups (under a suitable equivalence relation on $\overline{B}$).

**Proof** Here $\overline{A} = \{ A / A \text{ is a multi-anti fuzzy subgroup of } G \}$ and $\overline{B} = \{ A_t / A \in \overline{A}, \ t = (t_1, t_2, \ldots, t_i, \ldots), \ t_i \in [0,1], \forall i \}$

Let $H$ be any subgroup of $G$.

Claim: $\exists$ a 1-1 map between $H$ and $[A_i]$, where $[A_i]$ is the equivalence classes of $A_i$.

$\overline{B} = \overline{A} \times \mathbb{I}^n$, where $\mathbb{I}^n=[0,1] \times [0,1] \times \ldots \text{n times}$.

Define a relation ‘~’ on $\overline{B}$ by

$\forall (A,t), (B,s) \in \overline{B}, (A,t) \sim (B,s) \iff A_t = B_s$

Claim : ‘~’ is an equivalence relation on $\overline{B}$

i. Reflexive : Since $A_t = A_t$, $(A,t) \sim (A,t)$. Therefore, ‘~’ is reflexive.

ii. Symmetric: Suppose $(A,t) \sim (B,s)$.

$\Rightarrow A_t = B_s, \text{ since by ‘~’}$

$\Rightarrow B_s = A_t$

$\Rightarrow (B,s) \sim (A,t), \text{ since by ‘~’}$

Therefore, ‘~’ is symmetric.

iii. Transitive : Let $(A,r) \sim (B,s)$ and $(B,s) \sim (C,t)$.

Then this $\Rightarrow A_r = B_s$ and $B_s = C_t$

$\Rightarrow A_r = C_t$

$\Rightarrow (A,r) \sim (C,t)$

Therefore, ‘~’ is transitive.

Hence ‘~’ is an equivalence relation on $\overline{B}$.

So, the relation ‘~’ partitions $\overline{B}$.
Therefore, by the Theorem, each subgroup $H$ of $G$ can be realised as a multi-lower level subgroup of some fuzzy subgroup, $H = A_i$, \ldots \ldots \ldots (1)$, for some $A \in \mathcal{A}$ and $t=(t_1,t_2,\ldots)$, $t_i \in [0,1], \forall i$.

Define a map $f:[A_i] \to A_0(=H)$, since by (1)

Clearly, $f$ is one-one map.

Since $H$ is arbitrary subgroup of $G$, $f:[A_i] \to H$ is a one-one correspondence between the equivalence classes $[A_i]$ of $A$ and subgroups of $G$.

Hence the Theorem.

**Theorem 3.27** If $A$ is a multi-anti fuzzy subgroup of a group $G$ and $A(x^2)=A(x), \forall x \in G$, then $A(x)=A(e)$.

**Proof** It is clear.

### 4. Product Of Multi-Anti Fuzzy Subgroups

**Definition 4.1** Define the binary operation $\circ$ on $MFP(G)$, the multi-fuzzy power set of a group $G$ and the unary operation $^{-1}$ on $MFP(G)$ as follows:

$$\forall A,B \in MFP(G) \text{ and } \forall x \in G, \quad (A \circ B)(x) = \max \{ \min \{A(y), B(z) / y,z \in G, yz = x \} \}$$

and

$$A^{-1}(x) = A(x^{-1}).$$

We call $A \circ B$ the product of $A$ and $B$ and $A^{-1}$, the inverse of $A$. The binary operation $\circ$ is associative.

**Theorem 4.1** Let $A$ be a multi-fuzzy subset of a group $G$. Then $A$ is a multi-anti fuzzy subgroup of $G$ if $A$ satisfies the following conditions:

(i) $A \subseteq A \circ A$

(ii) $A^{-1} \subseteq A$ or $A^{-1} \supseteq A$ or $A^{-1} = A$

**Proof** Let $A$ be a multi-anti fuzzy subgroup of $G$.

Then $A(e) \leq A(x)$, $\forall x \in G$.

Claim (i): $A \subseteq A \circ A$

Since $A$ is a multi-anti fuzzy subgroup of $G$, $A(e) \leq A(x)$, $\forall x \in G$.

Let $x \in G$ be any arbitrary element of $G$.

Then $A(x) = A(xe)$

$$\leq \max \{ A(x), A(e) \} \text{, since } A \text{ is a multi-anti fuzzy subgroup of } G.$$

$$= \min \{ \max \{ A(x), A(e) \} \}$$

$$= \max \{ \min \{ A(x), A(e) \} \}$$

$$= \sup \{ \min \{ A(x), A(e) \} \}$$

$$= (A \circ A)(xe)$$
= (A◦A)(x)
That is, A(x) ≤ (A◦A)(x) , ∀x ∈ G
This implies A ⊆ A◦A
Claim (ii): A⁻¹ ⊆ A or A⁻¹ ⊇ A or A⁻¹ = A
A⁻¹(x) = (A⁻¹i(x))i∈I
        = (A⁻¹i(x⁻¹))i∈I , since A is a multi-anti fuzzy subgroup, each Aᵢ is anti fuzzy subgroup
        = (Aᵢ(x))i∈I
        = A(x) , ∀x ∈ G
Therefore, A⁻¹ = A and hence the Theorem.

**Theorem 4.2** Let A and B be any two arbitrary multi-anti fuzzy subgroups of a group G. Then A◦B is a multi-anti fuzzy subgroup of G ⇔ A◦B = B◦A
**Proof** Suppose A◦B is a multi-anti fuzzy subgroup of G
⇒ A◦B = (A◦B)⁻¹ , since by the Theorem 4.1.
⇒ A◦B = B⁻¹◦A⁻¹
⇒ A◦B = B◦A
Conversely, Suppose that A◦B = B◦A ...................(1)
Then ⇒ (A◦B)⁻¹ = (B◦A)⁻¹
⇒ (A◦B)⁻¹ = A⁻¹◦B⁻¹
⇒ (A◦B)⁻¹ = A◦B ....................(2)
Also, (A◦B) ◦ (A◦B) = A◦(B◦A)◦B , since ◦ is associative
= A◦(A◦B◦B) , since by (1)
= (A◦A)◦(B◦B) , since ◦ is associative
⊂ A◦B .........................(3),since A and B are multi-anti fuzzy subgroups,
(A◦A) ⊂ A and (B◦B) ⊂ B
Therefore, from (2) and (3), by the Theorem 4.1, (A◦B) is a multi-anti fuzzy subgroup of G.

**Theorem 4.3** A non-empty multi-fuzzy subset B of a multi-anti fuzzy subgroup A of a group G is a multi-anti fuzzy subgroup of A ⇔ B◦B⁻¹ ⊇ B.
**Proof** Let B be a multi-anti fuzzy subgroup of A.
Then this ⇒ B⁻¹ = B ..................(1)
Claim: B◦B⁻¹ ⊇ B
∀x ∈ G, B◦B⁻¹(x) = sup{min{B(y), B⁻¹(z)} / y, z ∈ G and yz = x}
\[ = \sup \{ \min \{ B(y), B(z) \} / y, z \in G \text{ and } yz = x \}, \text{ since by (1)} \]

\[ = B \cdot B(x) \]

\[ \trianglerightequal B(x), \text{ since } B \text{ is a multi-anti fuzzy subgroup} \]

Therefore, \( B \cdot B^{-1} \trianglerightequal B \)

Conversely, Suppose \( B \cdot B^{-1} \trianglerightequal B \) ………………(2)

Claim: \( B \) is a multi-anti fuzzy subgroup of \( A \)

That is, to prove that \( B(xy^{-1}) \leq \max \{ B(x), B(y) \}, \forall x, y \in A \)

Let \( x, y \in A \Rightarrow x, y \in G \), since \( A \) is a multi-anti fuzzy subgroup of \( G \).

Now, \( B(xy^{-1}) \leq B \cdot B^{-1}(xy^{-1}) \), by (2)

\[ = \sup \{ \min \{ B(x), B^{-1}(y^{-1}) \} \} \]

\[ = \min \{ \sup \{ B(x), B^{-1}(y^{-1}) \} \} \]

\[ \leq \sup \{ B(x), B^{-1}(y^{-1}) \} \]

\[ = \max \{ B(x), B(y^{-1}) \}, \text{ since } B = B^{-1} \text{ as } B \text{ is a multi-fuzzy set of } G. \]

\[ = \max \{ B(x), B(y) \}, \text{ since } B(x) = B(x^{-1}), \forall x \in G \text{ as } B \text{ is a multi-fuzzy set of } G. \]

That is, \( B(xy^{-1}) \leq \max \{ B(x), B(y) \}, \forall x, y \in A \).

Therefore, \( B \) is a multi-anti fuzzy subgroup of \( A \) and hence the Theorem.

**Theorem 4.4** A non-empty multi-fuzzy subset \( B \) of a multi-anti fuzzy subgroup \( A \) of a group \( G \) is a multi-anti fuzzy subgroup of \( A \) \( \iff \) \( B \cdot B^{-1} = B \)

**Proof** Let \( B \) be a multi-anti fuzzy subgroup of \( A \). Then \( B = B^{-1} \).

Therefore, \( B \cdot B^{-1} \trianglerightequal B \), by the Theorem 4.3…………………(1)

Claim : \( B \cdot B^{-1} \subseteq B \)

Now, \( \forall x \in G \),

\[ B(x) = \max \{ B(x), B(e) \}, B \text{ is a multi-anti fuzzy subgroup} \]

\[ \geq \min \{ \max \{ B(x), B(e) \} \} \]

\[ = \min \{ \max \{ B(x), B(e^{-1}) \} \} \]

\[ = \min \{ \max \{ B(x), B^{-1}(e) \} \} \]

\[ = \max \{ \min \{ B(x), B^{-1}(e) \} \} \]

\[ = \sup \{ \min \{ B(x), B^{-1}(e) \} \} \]

\[ = B \cdot B^{-1}(xe) \]

\[ = B \cdot B^{-1}(x) \]

Therefore, \( B(x) \geq B \cdot B^{-1}(x), \forall x \in G \)

That is, \( B \trianglerightequal B \cdot B^{-1} \) and hence the claim. ………………………………..(2)
From (1) and (2), \( B = B \circ B^{-1} \).

Conversely, Suppose \( B \circ B^{-1} = B \).

This implies that \( B \cdot B^{-1} \supseteq B \).

Therefore, by the Theorem 4.3, \( B \) is a multi-anti fuzzy subgroup of \( A \) and hence the Theorem.

**Theorem 4.5** Let \( A \) and \( B \) be any two multi-anti fuzzy subgroups of a group \( G \). Then \( A \circ B \) is a multi-anti fuzzy subgroup of \( G \) \( \iff \) \( A \circ B = B \circ A \).

**Proof**: Suppose \( A \circ B \) is a multi-anti fuzzy subgroup of \( G \).

\[
A \circ B = (A \circ B)^{-1}, \text{ since by the Theorem 4.1.}
\]

\[
A \circ B = B^{-1} \circ A^{-1}
\]

\[
A \circ B = B \circ A, \text{ since } A \text{ and } B \text{ are multi-anti fuzzy subgroups}
\]

Conversely, Suppose that \( A \circ B = B \circ A \).

To prove that \( A \circ B \) is a multi-anti fuzzy subgroup of \( G \), it is enough to prove that \( (A \circ B) \circ (A \circ B)^{-1} = A \circ B \).

Now, \( (A \circ B) \circ (A \circ B)^{-1} = (A \circ B) \circ (B^{-1} \circ A^{-1}) \)

\[
= A \circ (B \circ B^{-1}) \circ A^{-1}, \text{ since } \circ \text{ is associative}
\]

\[
= (A \circ B) \circ A^{-1}, \text{ since } B \circ B^{-1} = B \text{ as } B \text{ is a multi-anti fuzzy subgroup}
\]

\[
= (B \circ A) \circ A^{-1}, \text{ by the hypothesis}
\]

\[
= B \circ (A \circ A^{-1}), \text{ since } \circ \text{ is associative}
\]

\[
= B \circ A, \text{ since } A \circ A^{-1} = A \text{ as } A \text{ is a multi-anti fuzzy subgroup}
\]

\[
= A \circ B, \text{ since by the hypothesis}
\]

That is, \( (A \circ B) \circ (A \circ B)^{-1} = A \circ B \).

Therefore, \( A \circ B \) is a multi-anti fuzzy subgroup of \( G \) and hence the Theorem.

**Corollary**: If \( A \) and \( B \) are two multi-anti fuzzy subgroups of an abelian group \( G \), then \( A \circ B = B \circ A \) and hence by the Theorem 4.2, \( A \circ B \) is a multi-anti fuzzy subgroup of \( G \).

### 5. Fuzzification Of A Multi-Lower Level Subset

**Definition 5.1** Let \( A \) be a multi-fuzzy set of a set \( G \). Then the multi-lower level subset is \( A_t = \{ x \in G / A(x) \leq t \} \) where \( t = (t_1, t_2, \ldots) \), \( t_i \in [0,1], \forall i \). The fuzzification of \( A_t \) is the multi-fuzzy set \( \hat{A}_t \) defined by

\[
\hat{A}_t(x) = \begin{cases} A(x), & \text{if } x \in A_t \\ 0, & \text{otherwise} \end{cases}
\]

Clearly, \( \hat{A}_t \subseteq A \) and \( (\hat{A}_t)_t = A_t \).
Theorem 5.1 If $A$ is a multi-anti fuzzy subgroup of a group $G$, then $\hat{A}_t$ is also a multi-anti fuzzy subgroup of $G$, where $t = (t_1, t_2, \ldots), t_i \in [0,1], \forall i$ and $t \geq A(e)$.

**Proof** Clearly, $A_i$ is a subgroup of $G$.

Let $x,y \in G$.

Case (i): Suppose $x,y \in A_i$.

Then $\hat{A}_t(xy^{-1}) = A(xy^{-1})$, by the definition 5.1.

$\leq \max \{ A(x), A(y) \}$, since $A$ is multi-anti fuzzy subgroup.

$= \max \{ \hat{A}_t(x), \hat{A}_t(y) \}$, by the definition 5.1.

Hence, $xy^{-1} \in \hat{A}_t$.

Case (ii): Suppose $x \in A_i, y \notin A_i$. Then $xy^{-1} \notin A_i$.

$\hat{A}_t(xy^{-1}) = 0$, since by the definition 5.1.

$\leq \max \{ \hat{A}_t(x), \hat{A}_t(y) \}$

Hence, $xy^{-1} \notin \hat{A}_t$.

Case (iii): Suppose $x,y \notin A_i$. Then $xy^{-1} \notin A_i$.

$\hat{A}_t(xy^{-1}) = 0$, since by the definition 5.1.

$\leq \max \{ \hat{A}_t(x), \hat{A}_t(y) \}$

Hence, $xy^{-1} \in \hat{A}_t$.

Hence the Theorem.

6. Normal Multi-Anti Fuzzy Subgroup

**Definition 6.1** A multi-anti fuzzy subgroup $A$ of a group $G$ is called normal multi-anti fuzzy if $A(x) = A(y^{-1}xy), \forall x,y \in G$.

**Definition 6.2** [15] Let $A$ be a multi-fuzzy set of $G$. For $t=(t_1,t_2,\ldots,t_i,\ldots)$, where $t_i \in [0,1], \forall i$, the set $A_t = \{ x \in G / A(x) \leq t \}$ is called a multi-lower level subset of the multi-fuzzy set $A$.

**Definition 6.3** A multi-anti fuzzy subgroup $A$ of a group $G$ is called ‘normal multi-anti fuzzy’ (invariant multi-anti fuzzy) if $A(xy) = A(yx), \forall x,y \in G$.

**Lemma 6.1** Let $\mu$ be an anti fuzzy subgroup of a group $G$ and $g \in G$. Then $\mu(g) = t$ $\iff$ $g \notin \mu$ and $g \notin \mu_t$, $\forall t > s$, $t \in [0,1]$.

**Lemma 6.2** [19] Let $N$ be a normal subgroup of $G$. Then $xy \in N$ $\iff$ $yx \in N$, where $x,y \in G$. 
Lemma 6.3 [19] If $\mu$ is a normal fuzzy subgroup of $G$, then each level subgroup of $\mu$ is a normal subgroup of $G$.

Theorem 6.1 If $A$ is a normal multi-anti fuzzy subgroup of $G$, then each multi-lower level subgroup of $A$ is a normal subgroup of $G$.

Proof It is clear.

Lemma 6.4 [19] Let $\mu$ be a fuzzy subgroup of $G$. If $\mu_t, t \in \text{Image}\mu$, is a normal subgroup of $G$, then $\mu$ is normal fuzzy.

Theorem 6.2 Let $A$ be a multi-anti fuzzy subgroup of $G$. If $A_t, t \in \text{Image}A$, where $t=(t_1,t_2,\ldots,t_i,\ldots)$ is a normal subgroup of $G$, then $A$ is a normal multi-anti fuzzy subgroup of $G$.

Proof Let $x,y \in G$, $A(xy) = s$ and $r = (r_1,r_2,\ldots,r_i,\ldots)$, $s = (s_1,s_2,\ldots,s_i,\ldots)$, where $r_i, s_i \in [0,1]$ be such that $s > r$.

\[ \text{(i)} \]

Then by lemma 6.1, $xy \in A_s$ and $xy \notin A_r$.

Therefore, by lemma 6.2, $yx \in A_s$ and $yx \notin A_r$.

Since $yx \in A_s \Rightarrow A(yx) = s \quad \text{(ii)}$

Therefore, by (i) and (ii), $A(xy) = A(yx)$

Therefore, $A$ is a multi-anti fuzzy normal and hence the Theorem.

Theorem 6.3 A multi-anti fuzzy subgroup $A$ of $G$ is normal fuzzy $\iff$ each multi-lower level subgroups $A_t, t \in \text{Image}A$, are normal subgroup of $G$.

Proof By the Theorems 6.1 and 6.2, it is clear.

Theorem 6.4 Let $A, B \in \text{MFP}(G)$ and let $a= \max \{A(x) / x \in G\}$. Then the following assertions hold:

(i) $(A\circ B)(x) = \max \{ \min \{A(y), B(y^{-1}x) \} \} \quad \forall y \in G$

(ii) $(a \circ A)(x) = A(y^{-1}x), \forall x,y \in G$

(iii) $(A \circ a)(x) = A(xy^{-1}), \forall x,y \in G$
Proof It is clear.

**Theorem 6.5** Let $A \in \text{MFP}(G)$, the multi-fuzzy power set of a group $G$. Then the following assertions are equivalent:

(i) $A(yx) = A(xy)$, $\forall x, y \in G$; In this case, $A$ is called an *Abelian multi-fuzzy* subset of $G$.

(ii) $A(xy^{-1}) = A(y)$, $\forall x, y \in G$

(iii) $A(xy^{-1}) \geq A(y)$, $\forall x, y \in G$

(iv) $A(xy^{-1}) \leq A(y)$, $\forall x, y \in G$

(v) $A \circ B = B \circ A$, $\forall B \in \text{MFP}(G)$

**Proof** It is clear.

**Definition 6.4** Let $A$ be a multi-anti fuzzy subgroup of a group $G$. Then $A$ is called a normal multi-anti fuzzy subgroup of $G$ if $A$ is an Abelian multi-fuzzy subset of $G$. Let $\text{NMAF}(G)$ denote the set of all normal multi-anti fuzzy subgroups of $G$.

**Theorem 6.6** If $G$ is a commutative group, then every multi-anti fuzzy subgroup of $G$ is normal.

**Proof** Let $A$ be any multi-anti fuzzy subgroup of a commutative group $G$.

Claim: $A$ is normal. That is, $A(xy) = A(yx)$, $\forall x, y \in G$.

Let $x, y \in G$.

Since $G$ is a commutative group, $xy = yx$

$$\Rightarrow A(xy) = A(yx)$$

$$\Rightarrow A$$ is normal multi-anti fuzzy subgroup of $G$

Hence the Theorem.

**Theorem 6.7** A multi-anti fuzzy subgroup $A$ of a group $G$ is normal $\iff \forall x, y \in G$, $A(x) = A(yx^{-1})$.

**Proof** Let $A$ be a normal multi-anti fuzzy subgroup of $G$.

Then $A(xy) = A(yx)$ $\iff A((xy)y^{-1}) = A((yx)y^{-1})$

$\iff A(xe) = A(yxy^{-1})$

$\iff A(x) = A(yxy^{-1})$, $\forall x, y \in G$.

**Theorem 6.8** Let $A$ be a normal multi-anti fuzzy subgroup of a group $G$. Then $A_+ = \{ x \in G / A(x) = A(\epsilon) \}$ and $A^- = \{ x \in G / A(x) > 0 \}$, the Support of $A$, are normal multi-anti fuzzy subgroups of $G$.

**Proof** It is clear.
**Corollary:** The converse of the above Theorem 6.8 is not true.

**Proof** Let G be a group and H be a subgroup of G which is not normal.

Define the multi-fuzzy set A of G by $A(e) = 0$, $A(x) = 1/2$ if $x \in H \setminus \{e\}$ and $A(x) = 3/4$ if $x \in G \setminus H$.

Then A is a multi-anti fuzzy subgroup of G, since its level sets are subgroups of G.

Now, $A_e = H$ is not normal in G where $t = (\frac{1}{2}, \frac{1}{2}, \ldots)$.

Hence A is not a normal multi-anti fuzzy subgroup of G.

However, $A_e = \{e\}$ and $A^* = G$ are normal in G.

Hence the Corollary.

**Lemma 6.5** If A is a multi-anti fuzzy subgroup of a group G and H is any subgroup of G, then $A/H$ is a multi-anti fuzzy subgroup of H.

**Theorem 6.9** Suppose A is a multi-anti fuzzy subgroup of a group G. Let $N(A) = \{x \in G / A(xy) = A(yx), \forall y \in G\}$. Then N(A) is a subgroup of G and the restriction of A to N(A), $A/N(A)$, is a normal multi-anti fuzzy subgroup of N(A).

**Proof** Clearly, $e \in N(A)$. Therefore, $N(A) \neq \emptyset$.

Claim: N(A) is a subgroup.

Let $x, y \in N(A)$. For any $z \in G$, $A(xy^{-1}.z) = A(x.y^{-1}z)$, since by associative in G

\[
= A(y^{-1}z.x), \text{ since } x \in N(A)
\]
\[
= A((y^{-1}.z)x)^{-1}
\]
\[
= A(x^{-1}z^{-1}.y)
\]
\[
= A(y.x^{-1}z^{-1}), \text{ since } y \in N(A)
\]
\[
= A((y.x^{-1}z^{-1})^{-1})
\]
\[
= A(zxy^{-1})
\]

Therefore, $xy^{-1} \in N(A), \forall x, y \in N(A)$.

Hence N(A) is a subgroup of G.

Therefore, by the previous lemma 6.5, $A/N(A)$ is a multi-anti fuzzy subgroup of N(A).

Claim: Normal.

Let $x, y \in N(A)$. Then $A/N(A) (xy) = A/N(A) (yx)$

Therefore, $A/N(A)$ is a normal multi-anti fuzzy subgroup of N(A).

Hence the Theorem.
Remark: The subgroup \( N(A) \) of \( G \), defined in the above Theorem 6.9, is called the "normalizer of \( A \) in \( G \)".

Definition 6.5 If \( A \) and \( B \) are two multi-anti fuzzy subgroups of \( G \) and there exist \( u \in G \) such that \( A(x) = B(ux^{-1}), \forall x \in G \), then \( A \) and \( B \) are called conjugate multi-anti fuzzy subgroups (with respect to ‘u’) and we write \( A = B^u \) where \( B^u(x) = B(ux^{-1}), \forall x \in G \).

Theorem 6.10 Let \( A \) be a multi-anti fuzzy subgroup of a group \( G \). Then the cardinal number of the set \( \{ A^u / u \in G \} = [G : N(A)] \), the index of the normalizer \( N(A) \) in \( G \).

Proof Let \( u, v \in G \). Then
\[
A^u = A^v \iff A^u(x) = A^v(x), \forall x \in G
\]
\[
\iff A(ux^{-1}) = A(vx^{-1}), \forall x \in G, \text{ since by the definition 6.5}
\]
\[
\iff A(x) = A(x), \forall x \in G
\]
\[
\iff A(ux^{-1}.x) = A(x. uv^{-1}), \forall x \in G, \text{since by the Theorem 6.5 (i) as \( A \) is an abelian multi-fuzzy set of \( G \), equivalent conditions.}
\]
\[
\iff uv^{-1} \in N(A)
\]
\[
\iff u^{-1}N(A) = v^{-1}N(A)
\]

Thus, \( A^u \mapsto u^{-1}N(A) \) is a bijection map from \( \{ A^u / u \in G \} \) onto \( \{ uN(A) / u \in G \} \).

Hence the Theorem.

Theorem 6.11 Let \( A \) be a multi-anti fuzzy subgroup of a group \( G \). Then \( \cup A^u \in NMAF(G) \)
\( u \in G \)
and is the smallest normal multi-anti fuzzy subgroup of \( G \), that is contains \( A \).

Proof
Since \( A^u \in MAF(G), \forall u \in G, \cup A^u \in MAF(G) \).
\( u \in G \)

Claim: \( \cup A^u \) is normal.
\( u \in G \)

\( \forall x \in G, \{ A^u / u \in G \} = \{ A^{ux} / u \in G \} \)
Thus, \( \text{Max } A^{u}(xy^{-1}) = \text{Max } A( u(xy^{-1})u^{-1} ) \)
\( u \in G \quad u \in G \)
\[= \max_{u \in G} A(ux(ux)^{-1})\]

\[= \max_{u \in G} A^{ux}(y)\]

\[= \max_{u \in G} A^{u}(y), \forall x, y \in G.\]

Hence \(\cup A^{u} \in \text{NMAF}(G)\), by Theorem.

Now, let \(B \in \text{NMAF}(G)\) with \(B \supseteq A\).

Then \(\Rightarrow B^{u} \supseteq A^{u}, \forall u \in G.\)

\(\Rightarrow B = B^{u} \supseteq A^{u}, \forall u \in G.\)

\(\Rightarrow B \supseteq A^{u}, \forall u \in G.\)

\(\Rightarrow B \supseteq \cup A^{u}.\)

\(u \in G\)

Therefore, \(\cup A^{u}\) is the smallest normal multi-anti fuzzy subgroup of \(G\), that is, contains \(A.\)

\(u \in G\)

Hence the Theorem.

7. **References**


